

A Proof of The Lonely Runner Conjecture for Almost All Points

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1 The Lonely Runner Conjecture

Consider $k + 1$ runners on a circular track of circumference 1, and the vector

$$v = [v_1, v_2, \dots, v_{k+1}], v_i \in \mathbb{R}^+$$

Let $d(a, b) = \|a - b\|_1$

Conjecture: $\exists t_i \in \mathbb{R}^+$ such that $\min(d(x_i, x_j))_{i \neq j} \geq \frac{1}{k+1}$.

2 A Solution for Almost All Points

Suppose the components of v are linearly independent. Consider the family of discrete mappings $T_i : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. defined by

$$T_i(x) = \|x + \alpha * v_i\|$$

Where

$$\alpha \in (0, \frac{1}{\max v_i}) \setminus \mathbb{Q}, \frac{v_i}{\alpha} \notin \mathbb{N} \forall i.$$

Individually, each T_i represents the irrational rotation map on \mathbb{S}^1 with angle $\beta = \alpha * v_i$. The trajectory of x_i is dense over \mathbb{S}^1 . Further, the product of N such rotations is dense on the torus \mathbb{T}^k .

The density over \mathbb{T}^k forces that every possible configuration (x_1, x_2, \dots, x_k) is approximated arbitrarily well in finite time. Therefore,

$$\forall i, \forall \epsilon, \exists n_i \in \mathbb{N} : T_{j, j \neq i}^{n_i}(0) \in (-\epsilon, \epsilon), T_i^{n_i}(0) \in (.5 - \epsilon, .5 + \epsilon)$$

We have found a time, $t = n_i * \alpha$ where runner i is lonely. Any similar bound will be surpassed, as well.

The set of excluded points has Lebesgue measure zero. Consider that any rationally dependent velocity vector must be orthogonal to a different rational

vector. Therefore it lies in a hyperplane of dimension $n-1$. This hyperplane has measure zero in \mathbb{R}^n . As each such hyperplane is identified by a rational vector orthogonal to it, there are countably many. Thus the set of all such hyperplanes, which contains the set of rationally dependent vectors, is measure zero as well.

3 Coprime points

Consider the $n+1$ -runner Lonely Runner Conjecture, set one of the runners velocities to 0. Now observe that we need only consider the case of :

$$(1/D_1, \dots, 1/D_n)$$

where the D_i are pairwise relatively prime integers. (WLOG, $D_i < D_{i+1}$) Call such sets "best". Then we want a time T with $T/D_i = X_i \pmod{1}$, where the X_i are rational numbers between $1/(n+1), n/(n+1)$. If we choose the X_i so that $D_i X_i$ is an integer for all i , then we have that this is equivalent to: $T = X_i D_i \pmod{D_i}$ for all i . By the Chinese Remainder Theorem, we have

that there exists such a $T < \prod_{i=1}^n D_i$. (and moreover, T is an integer).

Now, if we have maps determined by the sets of initial velocities, and some time T , then the change of position of the n th runner due to the change in initial velocities is $\delta v * T$. So if we choose δv less than

$$(1/2 - 1/(n+1) - 1/2D_i) / \prod_{i=1}^n D_i$$

then the set of new initial velocities is also good. This requires some explanation. We choose the X_i to be the closest approximation to $1/2$ that the given D_i will permit. This is reasonable, because it keeps us the farthest from going out-of-bounds on the circle. The maximum error in approximating $1/2$ by a/D_i for some integer a , is $1/2D_i$. Thus, with the given δv , the maximum error incurred at the time T , when the original map was best, for the velocity D_i is:

$$((1/2 - 1/(n+1) - 1/2D_i) / \prod_{i=1}^n D_i) T < 1/2 - 1/(n+1) - 1/2D_i$$

The distance from $X_i D_i$ to $1/n$ or $n/(n+1)$, on the other hand, is at least:

$$1/2 - 1/(n+1) - 1/2D_i$$

by our above argument, so indeed, if the δv is as small as demanded above, then the new set of velocities is "good".

However, we are only concerned with the ratios between velocities. We will find the maximum and minimum velocity ratios of the good sets generated above

from the best sets. Any set that has all of its velocity ratios lying between the maximum and minimum for each of the i in $1, \dots, n-1$, must be good.

A velocity vector (x_1, \dots, x_n) is completely determined by the ratios

$$(x_2/x_1, x_3/x_2, \dots, x_n/x_{n-1})$$

We will now specify the *most* that each of these ratios can be for a good x-vector generated from a given best vector.

Consider the ratio x_{i+1}/x_i : The maximum velocity for the i th component $1/D_i$ is

$$1/(D_i - \delta v_i)$$

And the minimum for the D_{i+1} component is:

$$1/(D_{i+1} + \delta v_{i+1})$$

Then, if the i th ratio for some given x-vector is greater than

$$\frac{D_i - \delta v_i}{D_{i+1} + \delta v_{i+1}}$$

we are halfway there. It is easy to see by a similar calculation that if each ratio of the x-vector is less than

$$\frac{D_i + \delta v_i}{D_{i+1} - \delta v_{i+1}}$$

then it indeed lies in the appropriate ratio zone.

From this we can now reformulate the conjecture: A vector is good (with ratios (R_1, \dots, R_{n-1}) , all nonzero rational) if there exist pairwise relatively prime D_i $i=1, \dots, n-1$, with:

$$\frac{D_i - \delta v_i}{D_{i+1} + \delta v_{i+1}} < R_i < \frac{D_i + \delta v_i}{D_{i+1} - \delta v_{i+1}}$$

where

$$\delta v_i = ((1/2 - 1/(n+1) - 1/2D_i) / \prod_{i=1}^n D_i)$$

4 Small Strategy for this Style

Define the Quality $Q(D_i, D_{i+1})$ of an approximation D_i/D_{i+1} to R_i as the maximum that

$$\prod_{j=1}^n (D_j) / D_i D_{i+1}$$

may be and to still have:

$$\frac{D_i - \delta v_i}{D_{i+1} + \delta v_{i+1}} < R_i < \frac{D_i + \delta v_i}{D_{i+1} - \delta v_{i+1}}$$

It is a quick observation to see that if $\prod_{i=1}^n (D_j)/D_i D_{i+1} < Q(D_i, D_{i+1})$, for all i , then the ratio vector is good. I think this is probably the ideal way to go about this route, if to go about it at all.

Further, it can be shown that all the Q 's are almost completely determined once you have selected merely one of the D to do the approximating. Haven't gotten to that yet.

5 Problems

We will generate the volume of all such points formed by the quality approach. For a given vector $(P_1/D_1, \dots, P_n/D_n)$ the (projective) volume of the set of good velocities generated by this vector is the n Recall:

$$\frac{D_i - \delta v_i}{D_{i+1} + \delta v_{i+1}} < R_i < \frac{D_i + \delta v_i}{D_{i+1} - \delta v_{i+1}}$$

For all the good vectors (written as only ratios R_i). The volume is then merely the volume of the intervals of good R_i . These are:

$$\prod_{i=1}^{n-1} \left(\frac{D_i + \delta v_i}{D_{i+1} - \delta v_{i+1}} - \frac{D_i - \delta v_i}{D_{i+1} + \delta v_{i+1}} \right) = \frac{\prod_{i=1}^{n-1} 2(\delta v_{i+1} D_i - \delta v_i D_{i+1})}{\prod_{i=1}^{n-1} (D_{i+1} - \delta v_{i+1})(D_{i+1} + \delta v_{i+1})}$$

This then needs to be summed over all possible choices for P_i, D_i . We here order the fractions with the least D_i first, allowing negatives in the P_i only. We will index by the number of negatives in the velocity vector. Now, we have the following sum:

$$2^{n-1} \sum_{D_1, D_2, \dots, D_n \text{ coprime}} \sum_{P_1, \dots, P_n} \frac{\prod_{i=1}^{n-1} (\delta v_{i+1} D_i - \delta v_i D_{i+1})}{\prod_{i=1}^{n-1} (D_{i+1} - \delta v_{i+1})(D_{i+1} + \delta v_{i+1})}$$

6 Going About it Backwards

Here we will display an alternative approach to the Conjecture and provide a strategy for further efforts. Consider once again R^n , where one runner's velocity has been set to zero.

Some definitions are in order. Let $\|x\|$ for $x \in R$ be the least distance from $\|x\|$ to an integer. An "exact case" will be any velocity vector (v_1, \dots, v_n) in R^n that has

$$\sup_{t \in (0, \inf)} \inf_{i=1, \dots, n} (\|v_i * t\|) = 1/(n+1)$$

A "pseudo-exact case" will be any velocity vector, which is not an exact case, v_1, \dots, v_n in R^n that has

$$\sup_{t \in (0, T)} \inf_{i=1, \dots, n} (||v_i * t||) = 1/(n+1)$$

and

$$\sup_{t \in (0, T+\epsilon)} \inf_{i=1, \dots, n} (||v_i * t||) = 1/(n+1)$$

for some T , and some $\epsilon > 0$. First, observe that no rationally independent vector, say $V = (v_1, \dots, v_n)$, can be pseudo-exact. For then, we would need both $||v_i * T||$ and $||v_j * T||$ to be rational, for some i and j . Then v_i and v_j would be rational multiples of each other, a contradiction.

Suppose there exists some rational point in the velocity space $P = (p_1/q_1, \dots, p_n/q_n)$ for which the Conjecture does not hold. Then, for any rationally independent point $A = (a_1, \dots, a_n)$, let l denote the line from P to A . It is known that for A there exists some time T_1 for which $T(A)$ has each runner at least a distance $1/(n+1)$ from the start of the track, where $T : R^n \rightarrow R^n(v_1, v_2, \dots, v_n) \rightarrow (||v_1 * T||, \dots, ||v_n * T||)$ takes velocities to the position each of their runners has at time T . Define the map F_1 as follows:

$$F_1 = \inf_{t \in (0, T_1)} \sup_{i=1, \dots, n} (||v_i * t||)$$

Then $F_1(A) > 1/(n+1)$. We also verify that F_1 is continuous. Indeed, the sup and inf, taken on finite intervals, are continuous, and we merely have a composition of continuous functions. By hypothesis,

$$F_1(P) < 1/(n+1)$$

Thus, there exists some point B on l with $F_1(B) = 1/(n+1)$. Either it is exact, or it is pseudo-exact. However, this point cannot be rational (by construction